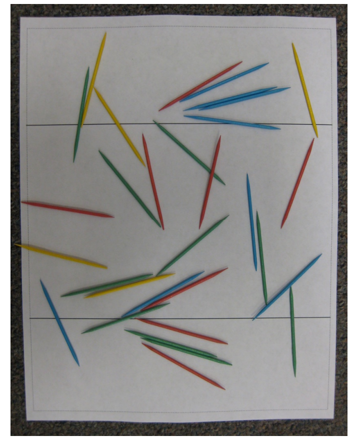
Throwing Pi

Needle your way through a pi toss.

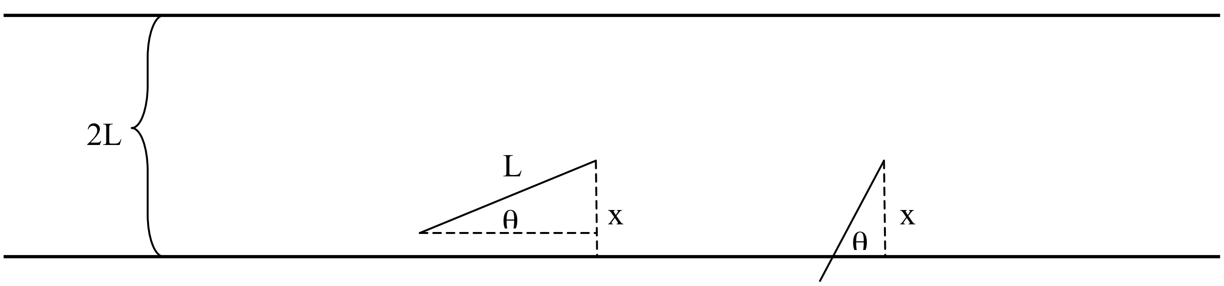


Use a classic problem of geometrical probability to find an important mathematical constant.

Materials and Preparation
toothpicks of the same size
paper
(optional) ruler, protractor, graph paper

To do and notice

1. Draw parallel lines on your paper so that the distance between them is twice the length of your toothpicks.
2. Randomly throw a large number of toothpicks onto the paper – the more the better.
3. Count the number of toothpicks that are touching a line.
4. Divide the total number of toothpicks that you threw by the number that touch a line. Does this number look familiar to you?
5. (optional) You can graphically determine the probability that a toothpick of length L will land on a line. You will need two coordinates to describe the position of a toothpick. Convenient ones to use are: the distance from the furthest edge of toothpick (that is still above the ine) to the line (x) and the angle between the toothpick and the line (θ). Due to the symmetry of the problem, you only have to consider $0<x<2L$ and $0<\theta<\pi/2$. Using a θ vs. x graph, plot all the possible coordinates that will result in the toothpick crossing a line. Divide the total area of possible toothpick positions, $2L*\pi/2$ by the area taken up by the plotted solutions. Is this a familiar number?

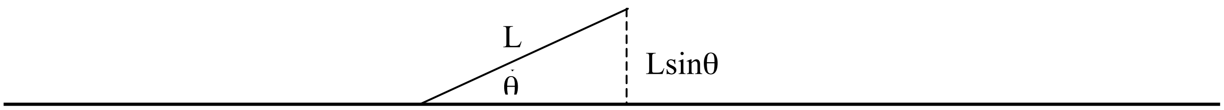


What's going on?

This activity explores a special case of Buffon's needle problem, first proposed in 1777 by Georges-Louis Leclerc, Comte de Buffon, which asks the probability of a needle hitting a line it is thrown at a group of evenly spaced parallel lines. In our case, the lines are spaced exactly twice the length (L) of the object. By physically throwing toothpicks, you can converge on the solution to this problem. If you throw enough toothpicks, and you are perfectly random about it, you'll find that the probability of hitting a line is 0.3183. If you convert this to a fraction, you may be surprised to find that it equals $1/\pi$! Since this number is hard to recognize, the activity asks you to find the reciprocal of the probability by dividing the total number of throws by the ones that hit a line. This number should be close to π, which is more familiar. If having π show up in this problem seems mysterious to you, read the math root below.

Math Root

By graphing the possible solutions and comparing the area of these solutions to the total area, you are actually doing calculus. If we use geometry to consider the possible mathematical solutions, we'll see that the toothpick will hit the line if $0<x<L\sin\theta$ for the angles $0<\theta<\pi/2$.



Using calculus, we can integrate to find the area of the solutions that cross and divide by the total area of possible solutions to find the probability of having a toothpick cross a line.

$$\frac{\text{solns that hit a line}}{\text{all possible solns}} = \frac{\int_0^{\frac{\pi}{2}} L\sin\theta\, d\theta}{2L\frac{\pi}{2}} = \frac{-L\cos\theta\Big|_0^{\frac{\pi}{2}}}{L\pi} = \frac{-L\left(\cos\frac{\pi}{2} - \cos 0\right)}{L\pi} = \frac{-L(0-1)}{L\pi} = \frac{1}{\pi}$$